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A SURVEY OF STAR PRODUCT GEOMETRY

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Abstract. A brief pedagogical survey of the star product is provided, through Groenewold's original construction based on the Weyl correspondence. It is then illustrated how simple Landau orbits in a constant magnetic field, through their Dirac Brackets, define a noncommutative structure since these brackets exponentiate to a star product—a circumstance rarely operative for generic Dirac Brackets. The geometric picture of the star product based on its Fourier representation kernel is utilized in the evaluation of chains of star products. The intuitive appreciation of their associativity and symmetries is thereby enhanced. This construction is compared and contrasted with the remarkable phase-space polygon construction of Almeida.

1. Introduction

The noncommutative star product of Groenewold [1] is the linchpin of deformation (phase-space) quantization [2, 3]. Currently, it is ubiquitous in matrix systems and in M-physics applications of non-commutative geometry ideas, such as in D-branes on a "magnetic" B-field background [4]. This product, connecting phase-space functions $f(x,p)$ and $g(x,p)$, is the unique associative pseudodifferential deformation [3] of ordinary products:

$$\star \equiv \; e^{i\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)/2} \; . \tag{1}$$

Since the star product involves exponentials of derivative operators, it may be evaluated in practice through translation of function arguments,

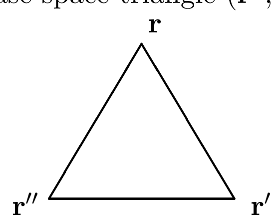
$$f(x,p) \star g(x,p) = f\left(x + \frac{i\hbar}{2}\overrightarrow{\partial}_p ,\; p - \frac{i\hbar}{2}\overrightarrow{\partial}_x\right)\; g(x,p). \tag{2}$$

When the phase-space functions involved consist of exponentials or simple polynomials, such star products amount to combinations of translations and finite-order PDEs, and allow first-principles solution of concrete deformation quantization problems [5]. However, for more complicated functions, explicit evaluations of long strings of star products in this language frequently appear intractable. (There exist well-developed numerical evaluation techniques [6], which, however, are not reviewed here.) What to do?

There may be a way out. The more practical Fourier representation of this product as an integral kernel has been utilized by Baker [7]:

$$f \star g = \frac{1}{\hbar^2 \pi^2} \int dp' dp'' dx' dx'' \; f(x', p') \; g(x'', p'') \\ \times \exp\left(\frac{-2i}{\hbar} \left(p(x' - x'') + p'(x'' - x) + p''(x - x') \right) \right). \tag{3}$$

The cyclic determinantal expression multiplying $-2i/\hbar$ in the exponent is twice the area of the phase-space triangle $(\mathbf{r}'', \mathbf{r}', \mathbf{r})$, where $\mathbf{r} \equiv (x, p)$,



namely,

$$2A(r'', r', r) = (\mathbf{r}' - \mathbf{r}) \wedge (\mathbf{r} - \mathbf{r}'') = \mathbf{r}'' \wedge \mathbf{r}' + \mathbf{r}' \wedge \mathbf{r} + \mathbf{r} \wedge \mathbf{r}''. \tag{4}$$

For example, in this representation, it is straightforward to work out the distinctive hyperbolic tangent composition law of phase-space Gaussians,

$$\exp\left(-\frac{a}{\hbar}(x^2 + p^2) \right) \; \star \; \exp\left(-\frac{b}{\hbar}(x^2 + p^2) \right) \\ = \frac{1}{1 + ab} \exp\left(-\frac{a + b}{\hbar(1 + ab)}(x^2 + p^2) \right), \tag{5}$$

which codifies the time evolution of the harmonic oscillator [3].

In this representation, multiple star turn out to be simpler to evaluate, and the geometrical constructions they motivate exhibit conspicuously the symmetries and the associativity of these products. The representation thus rises to the level of a ‘picture’, in Dirac’s sense of a “way of looking at the fundamental laws which makes their self-consistency obvious” [8]. Such evaluations are illustrated below, with some practical hints, for the standard

star product, as well as for some common variants and extensions, such as the supersymmetrized version. This survey is based on ref [9], but further covers alternate schemes and background.

2. Brief Historical Review—to be skipped by experts

To give the general reader a flavor of how the star product is defined in physics, some of the essentials of phase-space quantization which rely on it are reviewed briefly.

Weyl [10] introduced an association rule mapping invertibly c-number phase-space functions $f(x,p)$ (called classical kernels) to operators $\mathfrak{F}$ in a given ordering prescription. Specifically, $p \mapsto \mathfrak{p}$, $x \mapsto \mathfrak{x}$, and, in general,

$$\mathfrak{F}(\mathfrak{x},\mathfrak{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \ f(x,p) \exp(i\tau(\mathfrak{p}-p) + i\sigma(\mathfrak{x}-x)). \qquad (6)$$

The eponymous ordering prescription requires that an arbitrary operator, regarded as a power series in $\mathfrak{x}$ and $\mathfrak{p}$, be first ordered in a completely symmetrized expression in $\mathfrak{x}$ and $\mathfrak{p}$, by use of Heisenberg's commutation relations, $[\mathfrak{x},\mathfrak{p}] = i\hbar$. A term with m powers of $\mathfrak{p}$ and n powers of $\mathfrak{x}$ will be obtained from the coefficient of $\tau^m \sigma^n$ in the expansion of $(\tau\mathfrak{p} + \sigma\mathfrak{x})^{m+n}$. It is evident how the map yields a Weyl-ordered operator from a polynomial classical kernel. It includes every possible ordering with multiplicity one, e.g.,

$$6p^2x^2 \mapsto \mathfrak{p}^2\mathfrak{x}^2 + \mathfrak{x}^2\mathfrak{p}^2 + \mathfrak{p}\mathfrak{x}\mathfrak{p}\mathfrak{x} + \mathfrak{p}\mathfrak{x}^2\mathfrak{p} + \mathfrak{x}\mathfrak{p}\mathfrak{x}\mathfrak{p} + \mathfrak{x}\mathfrak{p}^2\mathfrak{x} \ . \qquad (7)$$

Weyl-ordered operators clearly close among themselves under operator multiplication, given the degenerate Campbell-Baker-Hausdorff identity. In a study of the uniqueness of the Schrödinger representation, von Neumann [11] adumbrated the composition rule of classical kernels in an operator product, appreciating that Weyl's correspondence was in fact a homomorphism. (Effectively, he arrived at a convolution representation of the star product.) Finally, Groenewold [1] neatly worked out in detail how the classical kernels f and g of two operators $\mathfrak{F}$ and $\mathfrak{G}$ must compose to yield the classical kernel of $\mathfrak{F}\mathfrak{G}$,

$$\begin{aligned} \mathfrak{F}\mathfrak{G} &= \frac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x',p') g(x'',p'') \\ &\times \exp i(\xi(\mathfrak{p}-p') + \eta(\mathfrak{x}-x')) \exp i(\xi'(\mathfrak{p}-p'') + \eta'(\mathfrak{x}-x'')) \qquad (8) \end{aligned}$$

$$= \frac{1}{(2\pi)^4} \int d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x',p') g(x'',p'') \exp i\left((\xi+\xi')\mathfrak{p} + (\eta+\eta')\mathfrak{x}\right)$$

$$\times \ \exp i\left(-\xi p' - \eta x' - \xi' p'' - \eta' x'' + \frac{\hbar}{2}(\xi\eta' - \eta\xi')\right).$$

Changing integration variables to

$$\xi' \equiv \frac{2}{\hbar}(x-x'), \quad \xi \equiv \tau-\frac{2}{\hbar}(x-x'), \quad \eta' \equiv \frac{2}{\hbar}(p'-p), \quad \eta \equiv \sigma-\frac{2}{\hbar}(p'-p), \tag{9}$$

reduces the above integral to

$$\mathfrak{F}\mathfrak{G} = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp \;\; \exp i\left(\tau(\mathfrak{p}-p)+\sigma(\mathfrak{x}-x)\right) \;\; (f\star g)(x,p), \tag{10}$$

where $f \star g$ is the expression (3). This is the original [1], and still most physically compelling definition of the star product. It is this fundamental isomorphism of operator products to associative strings of star multiplications which enables the formulation of Quantum Mechanics in phase space [2, 3].

Remark. On a phase-space torus, $x+2\pi \equiv x$, $p+2\pi \equiv p$, the integer modes of periodic functions, $f(x,p) = \exp i(m'x+n'p)$, $g(x,p) = \exp i(m''x+n''p)$, compose simply under the star product,

$$f \star g = e^{-i\hbar(m'n''-m''n')/2} \;\; \exp i((m'+m'')x + (n'+n'')p). \tag{11}$$

Thus, in this maximally graded basis, the antisymmetrization of the star product (Moyal's commutator) has a trigonometric structure constant [12]: $\sin \hbar(m'n''-m''n')/2$. Its argument involves a cross product of two-dimensional integer-valued vectors. For $\hbar = 4\pi/N$, this Lie algebra is identifiable [12] with $SU(N)$ for N odd, and $SU(N/2)$ for N even, and thus with $SU(\infty)$ as $\hbar \to 0$.

3. A remark on noncommutativity and Dirac Brackets

Parenthetically, the connection of this simple phase space to D-brane noncommutativity, a current application, arises as follows. The archetype of noncommutative geometry induced by magnetic flux is classical Landau orbital motion of a massless particle on a plane, in a constant magnetic field background. Thus, for a vector potential $A^i = x^j \epsilon^{ji} B/2$, and suppressing the kinetic term (which may be done consistently [13]), and morevoer choosing, e.g., a harmonic scalar potential [13]:

$$L = \frac{B}{2} x^i \epsilon^{ij} \frac{dx^j}{dt} - \frac{k}{2}\mathbf{x}^2, \tag{12}$$

whence the canonical momenta are constrained to the respective transverse coordinates,

$$p^i = -\frac{B}{2}\epsilon^{ij} x^j. \tag{13}$$

This amounts to two second class constraints, so the Poisson Brackets $\{x^i, p^j\} = \delta^{ij}$, $\{x^i, x^j\} = 0$, and $\{p^i, p^j\} = 0$ must be upgraded to Dirac Brackets, instead, for consistency:

$$\begin{aligned}
\{\!|x^i, x^j|\!\} &= \{x^i, x^j\} - \{x^i,\ p^k + \frac{B}{2}\epsilon^{kl}x^l\}\frac{\epsilon^{mk}}{B}\{p^m + \frac{B}{2}\epsilon^{mn}x^n,\ x^j\} \\
&= -\frac{\epsilon^{ij}}{B}\ ,
\end{aligned} \tag{14}$$

and likewise,

$$\{\!|x^i, p^j|\!\} = \frac{\delta^{ij}}{2}, \qquad \{\!|p^i, p^j|\!\} = -\frac{B}{4}\epsilon^{ij}\ . \tag{15}$$

For these particular expressions to hold, it is crucial that the magnetic field B be a constant.

Thus, the Hamiltonian that results from the Lagrangian (which is linear in the velocity), $H = \frac{k}{2}\mathbf{x}^2$, yields the correct equations of motion describing the cyclotron orbits,

$$\frac{dx^i}{dt} = \{\!|x^i, H|\!\} = -\frac{k}{B}\epsilon^{ij}x^j. \tag{16}$$

The two directions x and y, then, do not commute,

$$\{\!|x, y|\!\} = -\frac{1}{B}\ , \tag{17}$$

so that perpendicular directions behave as canonical momenta to each other.

Consequently, such a plane maps to the elementary phase space used for illustration in this talk, and the Dirac Bracket maps to the Moyal Bracket (the antisymmetrization of the star product already mentioned).

The reason these Dirac Brackets exponentiate effortlessly to produce an associative star product is because the constraints considered are linear, and hence these Dirac Brackets specify a Poisson manifold with *constant* bracket kernel:

$$\begin{aligned}
\{\!|f, g|\!\} = f\ (\overleftarrow{\partial}_{x_i}\ \frac{1}{2}\ \overrightarrow{\partial}_{p_i} - \overleftarrow{\partial}_{p_i}\ \frac{1}{2}\ \overrightarrow{\partial}_{x_i} - \overleftarrow{\partial}_{x_i}\ \frac{\epsilon^{ij}}{B}\ \overrightarrow{\partial}_{x_j} - \overleftarrow{\partial}_{p_i}\ \frac{\epsilon^{ij}B}{4}\ \overrightarrow{\partial}_{p_j})\ g \\
\equiv \partial_i f\ J_{ij}\ \partial_j g\ .
\end{aligned} \tag{18}$$

In the example considered, the space where the antisymmetric matrix J_{ij} acts is 4-dimensional ($z_{2i-i} = x_i$, $z_{2i} = p_i$).

Because J_{ij} here is constant, the system is one linear transformation away from a reduced standard phase space of *one* x, p pair,

$x = x_1 + (2/B)p_2$, $p = p_1/2 - (B/4)x_2$, effectively governed by Poisson (not Dirac) Brackets, which exponentiate associatively (e.g., see the next section). The exponential

$$e^{i\hbar \overleftarrow{\partial}_i J_{ij} \overrightarrow{\partial}_j} \tag{19}$$

is manifestly associative and hence defines a good star product. This is the type of star product exploited in the current literature on non-commutative applications to M-theory.

In sharp contrast, for *nonlinear* constraints, i.e. *nonconstant* $J_{ij}(\mathbf{x}, \mathbf{p})$, exponentiation of the relevant Dirac Bracket kernel does *not* yield an associative star product, in general. (This is illustrated for hyperspherical phase space in [14], with brackets

$$\{\!\!\{x_i, x_j\}\!\!\} = 0, \qquad \{\!\!\{x_i, p_j\}\!\!\} = \delta_{ij} - x_i x_j, \qquad \{\!\!\{p_i, p_j\}\!\!\} = x_j p_i - x_i p_j \ , \tag{20}$$

and whence bracket kernel

$$\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_x \cdot xx \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_x + \overleftarrow{\partial}_p \cdot xx \cdot \overrightarrow{\partial}_x + \overleftarrow{\partial}_p \cdot px \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot xp \cdot \overrightarrow{\partial}_p \ . \tag{21}$$

It is at variance with the generally *non*-associative proposal of [15].) That is, even though the Dirac Bracket satisfies the Jacobi identity [16], in this language,

$$(\partial_k J_{ij}) J_{kl} + (\partial_k J_{jl}) J_{ki} + (\partial_k J_{li}) J_{kj} = 0, \tag{22}$$

nevertheless, the exponential $\exp\left(i\hbar \overleftarrow{\partial}_i J_{ij} \overrightarrow{\partial}_j\right)$ itself fails associativity, in general.

Kontsevich [17] discovered, instead, elaborate graphical rules for the generation of the appropriate associative star product as a series in $\hbar$. The series starts as

$$\begin{aligned} &1 + i\hbar \overleftarrow{\partial}_i J_{ij} \overrightarrow{\partial}_j - \frac{\hbar^2}{2} \left(\overleftarrow{\partial}_i \overleftarrow{\partial}_k J_{ij} J_{kl} \overrightarrow{\partial}_j \overrightarrow{\partial}_l \right) \\ &- \frac{\hbar^2}{3} \left(\overleftarrow{\partial}_i \overleftarrow{\partial}_k J_{ij} (\partial_j J_{kl}) \overrightarrow{\partial}_l - \overleftarrow{\partial}_k J_{ij} (\partial_j J_{kl}) \overrightarrow{\partial}_i \overrightarrow{\partial}_l \right) + O(\hbar^3). \end{aligned} \tag{23}$$

The departure from the exponential is apparent in the second $O(\hbar^2)$ term.

4. Composition of star products

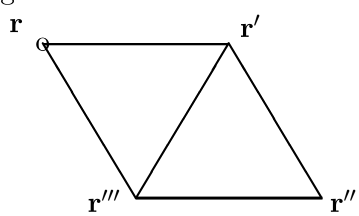
In the Fourier representation, a triple star product can be expressed relatively simply [9],

$$(f \star g) \star h = \frac{1}{\hbar^4 \pi^4} \int d\overline{p} dp' dp'' dp''' d\overline{x} dx' dx'' dx''' f(x', p') g(x'', p'') h(x''', p''')$$
$$\times \exp \frac{-4i}{\hbar} \left(A(r'', r', \overline{r}) + A(r''', \overline{r}, r) \right) . \tag{24}$$

Fortunately, the intermediate $d\overline{x}\ d\overline{p}$ integrations collapse to mere δ-functions:

$$(f \star g \star h)(x,p) = \frac{1}{\hbar^2 \pi^2} \int dp' dp'' dp''' dx' dx'' dx''' f(x', p') g(x'', p'') h(x''', p''')$$
$$\times \delta(x - x' + x'' - x''') \delta(p - p' + p'' - p''') \ \exp \left(\frac{-4i}{\hbar} A(r''', r'', r') \right) . \tag{25}$$

The product thus hinges on a triangle whose area enters in the phase of the exponential. The effective phase-space argument $\mathbf{r} = (x, p)$ of the product is now rigidly constrained: it lies on the new vertex of the parallelogram resulting from doubling up the triangle $(\mathbf{r}''', \mathbf{r}'', \mathbf{r}')$, such that $\mathbf{r}' - \mathbf{r}'''$ is one diagonal; the argument $\mathbf{r}$ lies at the end of the *other* diagonal, across $\mathbf{r}''$,



It is then straightforward to note how this expression bears no memory of the grouping (order of association) in which the two $\star$-multiplications were performed, since the vertex $\mathbf{r}$ of the parallelogram is reached from $\mathbf{r}'''$ by translating through $\mathbf{r}' - \mathbf{r}''$, or, equivalently, from $\mathbf{r}'$ by translating through $\mathbf{r}''' - \mathbf{r}''$. As a result [9], this may well realize the briefest graphic proof of the distinctive associativity property of the star product,

$$(f \star g) \star h = f \star (g \star h) \ . \tag{26}$$

The symmetries of the triple star product, (1-3 complex conjugacy; cyclicity in the phase and alternating cyclicity in the effective argument; etc.) are now evident by inspection. E.g., for $f = h$, the triple product is real.

Moreover, integration of this triple product with respect to its argument $\mathbf{r}$ (tracing), e.g. to yield a lagrangian interaction term, trivially eliminates the δ-function to result in a compact cyclic expression of the above triangle construction for the three functions star-multiplied,

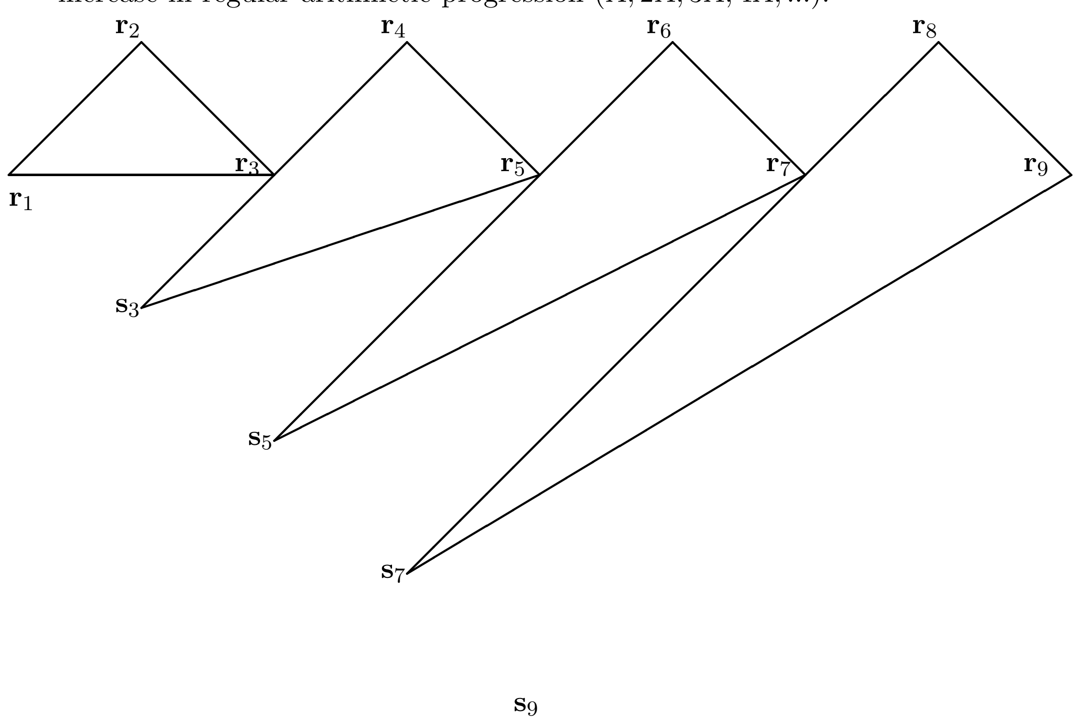
$$\int dx dp \ f \star g \star h = \frac{1}{\hbar^2 \pi^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 f(\mathbf{r}_1) g(\mathbf{r}_2) h(\mathbf{r}_3) \ \exp \left(\frac{-4i}{\hbar} A(r_3, r_2, r_1) \right) . \tag{27}$$

A four function star product (with three stars) involves the sum of the areas of two triangles, $(\mathbf{r}_3, \mathbf{r}_2, \mathbf{r}_1)$ and $(\mathbf{r}, \mathbf{r}_4, \mathbf{r}_1 - \mathbf{r}_2 + \mathbf{r}_3)$. A five function star product involves the exponential of the sum of areas of two triangles, $(\mathbf{r}_3, \mathbf{r}_2, \mathbf{r}_1)$ and $(\mathbf{r}_5, \mathbf{r}_4, \mathbf{r}_1 - \mathbf{r}_2 + \mathbf{r}_3)$, with the effective argument restricted by $\delta(\mathbf{r} - \mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 + \mathbf{r}_4 - \mathbf{r}_5) \equiv \delta(\mathbf{r} - \mathbf{s}_5)$.

Recursively, so on for even numbers of $\star$-multiplied functions, the phase involving the sums $A(\mathbf{r}_3, \mathbf{r}_2, \mathbf{r}_1) + A(\mathbf{r}_5, \mathbf{r}_4, \mathbf{s}_3) + ... + A(\mathbf{r}, \mathbf{r}_{2n}, \mathbf{s}_{2n-1})$. Note the cyclic symmetry, $\mathbf{r}_1 \mapsto \mathbf{r}_2 \mapsto \cdots \mathbf{r}_{2n} \mapsto \mathbf{r} \mapsto \mathbf{r}_1$.

Respectively, for odd numbers of functions, the phase involves sums $A(\mathbf{r}_3, \mathbf{r}_2, \mathbf{r}_1) + A(\mathbf{r}_5, \mathbf{r}_4, \mathbf{s}_3) + ... + A(\mathbf{r}_{2n+1}, \mathbf{r}_{2n}, \mathbf{s}_{2n-1})$, while the effective phase-space argument is restricted to $\mathbf{r} = \mathbf{s}_{2n+1} \equiv \sum_{m=1}^{2n+1} (-)^{m+1} \mathbf{r}_m$. Note the cyclic symmetry in the phase, again, and the alternating cyclic structure in the effective phase-space argument.

As an illustration, consider phase-space points $\mathbf{r}_i$ arrayed in a regular zigzag pattern, (i.e. for the $\star$-multiplied functions getting support only on those points on the zigzag). The arguments of the δ-functions, $\mathbf{s}_{2n+1}$, then lie on a line, while the areas of the triangles demarcated by these points increase in regular arithmetic progression $(A, 2A, 3A, 4A, ...)$:



This result is independent of the pitch of the zigzag, i.e. the angle at $\mathbf{r}_2$ —which, in this figure, is chosen to be $\pi/2$, since this is a local maximum of the areas A of the triangles for variable pitch but fixed lengths

$\mathbf{r}_i - \mathbf{r}_{i+1}$. One might well wonder if the configuration pictured could be used to define a "classical path": its contribution to the phase of the exponential through the sum of all triangle areas, $(1+2+3+4+...)A$, is stationary with respect to variations such as this angle variation discussed. The question suggests itself, then, whether configurations stationary under *all* variations can be constructed, leading to a stationary phase evaluation of large/infinite star products, e.g. useful in evaluating $\star$-exponentials (which yield time-evolution operators in phase-space [3]); but, so far, no cogent general answers appear at hand[1].

5. Almeida's Polygon

After completion of ref [1], an alternate, intriguing, earlier geometrical insight on star products was brought to my attention, [18], which organizes the problem into a different construction. In the conventions employed above, it essentially demonstrates the following [18]:

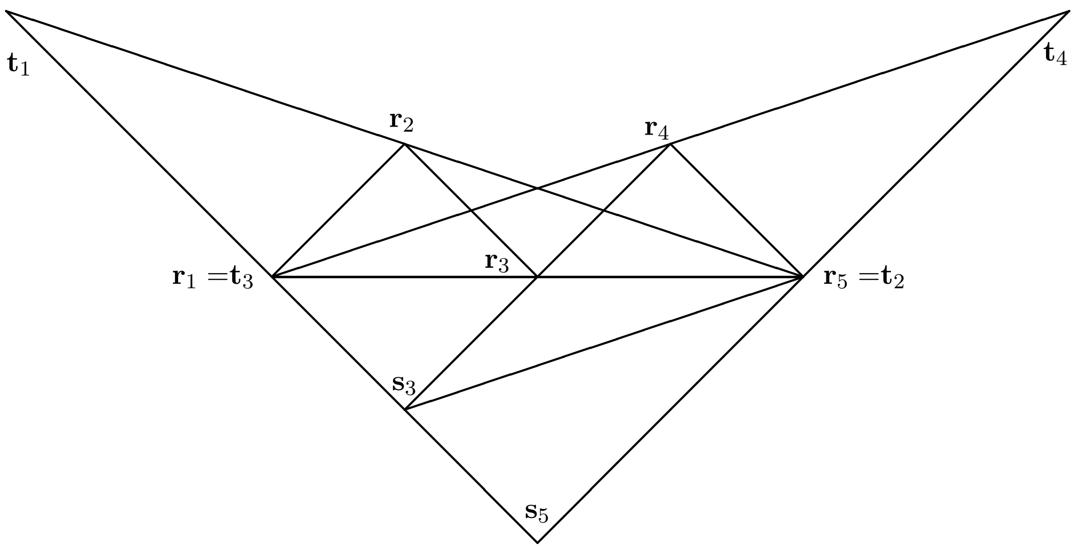
> For a star product of an odd number $(2n+1)$ of functions, a unique $2n+1$-gon is propounded in phase space, whose area equals 4 times the sum of the areas of triangles considered above $(A(\mathbf{r}_3, \mathbf{r}_2, \mathbf{r}_1) + ... + A(\mathbf{r}_{2n+1}, \mathbf{r}_{2n}, \mathbf{s}_{2n-1}))$ in the phase of the exponential kernel. Almeida's polygon is constructed as follows.
>
> First, $\mathbf{s}_{2n+1}$ (specified above) is constructed simply by alternating vector addition. One then extends the segment $\mathbf{s}_{2n+1} - \mathbf{r}_1$ by an equal length past $\mathbf{r}_1$, to a point denoted by $\mathbf{t}_1$; thus, $\mathbf{r}_1$ lies at the midpoint of the first side of the polygon, $\mathbf{s}_{2n+1} - \mathbf{t}_1$. Likewise, from $\mathbf{t}_1$, one connects to $\mathbf{r}_2$, then extending to $\mathbf{t}_2$, such that $\mathbf{r}_2$ lies at the midpoint of the second side, $\mathbf{t}_2 - \mathbf{t}_1$. So on, to $\mathbf{t}_{2n}$, whence one joins $\mathbf{t}_{2n}$ to $\mathbf{s}_{2n+1}$. It can be seen that $\mathbf{r}_{2n+1}$ lies on $\mathbf{t}_{2n} - \mathbf{s}_{2n+1}$, and, in fact, at its midpoint.
>
> The polygon constructed has $\mathbf{r}_1, ...\mathbf{r}_{2n+1}$ at the midpoints of its sides.

This polygon construction extends to an even number of functions $\star$-multiplied, with the effective argument serving as the $2n+1$th point, as before. For nonconvex polygons, the signed sum of convex segments must be considered for the total area.

This polygon is completely "egalitarian", in that no order of association is even apparent, which highlights its abstract elegance—whether it appears more economical in practice than the sequence of triangles considered in the

[1]As an amusing curiosity, one may consider four phase-space functions, $a(\mathbf{r}')$, $b(\mathbf{r}'')$, $c(\mathbf{r}''')$, $d(\mathbf{r})$, supported only at the vertices of the parallelogram $(\mathbf{r}, \mathbf{r}''', \mathbf{r}'', \mathbf{r}')$, displayed after eqn (17). A product $a \star (b \star c \star a \star b \star d \star a \star c \star d)$ then repeats itself in a "limit cycle", as additional octuple product factors $\star(b \star c \star a \star b \star d \star a \star c \star d)$ are appended to the right of the product, each such factor contributing to the phase twice the area of the parallelogram.

previous section, or not. Illustrated in the above configuration for points $\mathbf{r}_1, ..., \mathbf{r}_5$, it turns out to be the nonconvex pentagon $(\mathbf{t}_4, \mathbf{t}_3(=\mathbf{r}_1),\ \mathbf{t}_2(=\mathbf{r}_5),\ \mathbf{t}_1, \mathbf{s}_5)$:



This Almeida pentagon then has area $A(\mathbf{t}_2, \mathbf{t}_1, \mathbf{s}_5) + A(\mathbf{t}_4, \mathbf{t}_3, \mathbf{t}_2)$, which, indeed, amounts to 4 times the sum of areas of the two triangles of the recursive construction of the previous section, $A(\mathbf{r}_3, \mathbf{r}_2, \mathbf{r}_1) + A(\mathbf{s}_3, \mathbf{r}_5, \mathbf{r}_4)$.

The simple sequence of triangles of the assemblage of the previous section assumes a given order of association (grouping)—but associativity has already been demonstrated. On the other hand, by its recursiveness, the addition to the existing assembly of more phase-space points, e.g. $\mathbf{r}_6$ and $\mathbf{r}_7$ here, merely requires the evaluation of an extra triangle. By contrast, the corresponding Almeida heptagon is thoroughly different, as it starts from a different point, $\mathbf{s}_7$, so that the pentagon already evaluated is not of particular practical significance.

6. A Variant Product

A variant of the star product (cohomologically equivalent to it) is the lopsided associative product of Voros [19],

$$\rtimes\!\!\ltimes \;\equiv\; e^{i\hbar\overleftarrow{\partial}_x\overrightarrow{\partial}_p}\ . \tag{28}$$

It is sometimes convenient to rotate phase-space variables canonically (i.e. preserving their Poisson Brackets),

$$(x,p) \mapsto (\frac{x+ip}{\sqrt{-2i}}\ ,\frac{x-ip}{\sqrt{-2i}})\ , \tag{29}$$

to represent this product as

$$\begin{aligned} \bowtie \quad &\equiv \quad e^{\hbar(\overleftarrow{\partial}_x - i\overleftarrow{\partial}_p)(\overrightarrow{\partial}_x + i\overrightarrow{\partial}_p)/2} = e^{i\hbar(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x)/2}\ e^{i\hbar(\overleftarrow{\partial}_x\overrightarrow{\partial}_x + \overleftarrow{\partial}_p\overrightarrow{\partial}_p)/2} \\ &= \quad \star\ e^{-\hbar(\overleftarrow{\partial}_x^2 + \overleftarrow{\partial}_p^2)/4}\ e^{-\hbar(\overrightarrow{\partial}_x^2 + \overrightarrow{\partial}_p^2)/4}\ e^{\hbar((\overleftarrow{\partial}_x + \overrightarrow{\partial}_x)^2 + (\overleftarrow{\partial}_p + \overrightarrow{\partial}_p)^2)/4}\ . \end{aligned} \tag{30}$$

This turns out to be the covariant transform of the star product which controls the dynamics when Wigner distributions are transformed into Husimi distributions [20], a smoothed representation popular in applications.

It is plain that the Gaussian-Laplacian factors,

$$T^{-1}(\partial_x, \partial_p) \equiv \exp(-\hbar(\partial_x^2 + \partial_p^2)/4), \tag{31}$$

merely dress the standard star product into Voros' product [19],

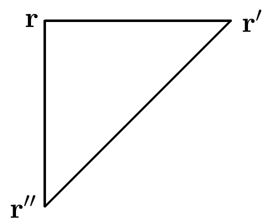
$$T(f \star g) = T(f) \bowtie T(g)\ . \tag{32}$$

The Lie algebra of brackets of $\bowtie$, i.e. the kernel of $f \bowtie g - g \bowtie f$, starts with the Poisson Brackets to $O(\hbar)$. By the above equivalence, this Lie algebra is seen to be equivalent to the Moyal algebra [2] (the algebra of brackets of $\star$, i.e. $\{\{f,g\}\} \equiv f \star g - g \star f$), in comportance with the general result on the essential uniqueness of the Moyal algebra as the one-parameter deformation of the Poisson Bracket algebra [21].

Actually, in Fourier space, this product in its original representation (28) appears even simpler than the star product,

$$\begin{aligned} (f \bowtie g)(x,p) = \frac{1}{2\pi\hbar} \int d\mathbf{r}' d\mathbf{r}''\ f(x',p')\ g(x'',p'')\ \delta(x''-x)\delta(p'-p) \\ \times \exp\left(\frac{i}{\hbar}(x''-x')(p'-p'')\right). \end{aligned} \tag{33}$$

The phase-space integral is then effectively a two-dimensional $\int dx' dp''$, not a four-dimensional one, as the kernel has vanishing support everywhere but on the lines $x'' = x$, $p' = p$. The triangle whose doubled area multiplies $-i/\hbar$ in the exponent is now a phase-space *right* triangle $(\mathbf{r}'', \mathbf{r}', \mathbf{r})$, with its side $\mathbf{r} - \mathbf{r}'$ horizontal, and its side $\mathbf{r} - \mathbf{r}''$ vertical:



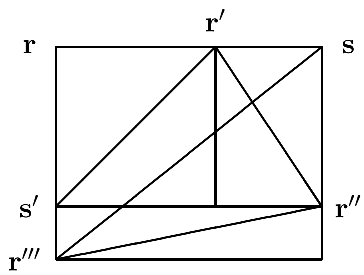
The triple product is then seen to be actuating shifts on a rectangular lattice,

$$(f \bowtie g) \bowtie h = \frac{1}{(2\pi\hbar)^2} \int d\mathbf{r}' d\mathbf{r}'' d\mathbf{r}''' f(x', p') g(x'', p'') h(x''', p''') \times \tag{34}$$

$$\times \delta(x''' - x) \delta(p' - p) \;\exp\left(\frac{i}{\hbar} (x'(p'' - p') + x''(p''' - p'') + x'''(p' - p''')) \right).$$

The phase is a cyclic expression with no memory of the order of association, which thus proves associativity for this product, $(f \bowtie g) \bowtie h = f \bowtie (g \bowtie h)$.

Pictorially, the phase is the area of the entire encompassing rectangle with diagonal $\mathbf{r}''' - \mathbf{s}$, minus the area of the rectangle with diagonal $\mathbf{r}' - \mathbf{r}''$; which is also equal to the *sum* of the areas of the rectangles with diagonals $\mathbf{s}' - \mathbf{r}'$, and $\mathbf{r}''' - \mathbf{r}''$, respectively. (In general, it is not twice the area of the triangle $(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')$.)



The construction for an n-tuple $\bowtie$-product follows simply,

$$\frac{1}{(2\pi\hbar)^n} \int d\mathbf{r}_1 ... d\mathbf{r}_n \; f_1(\mathbf{r}_1) ... f_n(\mathbf{r}_n) \; \delta(x_n - x) \delta(p_1 - p)$$

$$\times \exp\left(\frac{i}{\hbar} \sum_{m=1}^{n} x_m (p_{m+1} - p_m) \right), \tag{35}$$

where p_{n+1} is defined as p_1. A fleeting inspection of this formula suggests an effective nearest-neighbor interaction in a natural chain.

Remark. More recondite star products for particular nonflat phase-space manifolds, including Kähler manifolds, can also be formulated through integral kernels involving the Calabi function [22].

7. Graded Extension

A superspace generalization of the star-product was introduced in ref [12], (to codify the graded extension of Moyal's algebra introduced in ref [23]),

$$(1+\hbar \overleftarrow{\partial}_\theta \overrightarrow{\partial}_\theta) \ \star \ \equiv \ \diamond \ \star \ . \tag{36}$$

Here, θ is the superspace Grassmann variable (nilpotent, and commuting with the phase-space variables): the extended star-product is then a direct product of the conventional piece with a superspace factor $1+\hbar \ \overleftarrow{\partial}_\theta \overrightarrow{\partial}_\theta$. Thus, the above extended product could have been alternatively written as

$$e^{\hbar \overleftarrow{\partial}_\theta \overrightarrow{\partial}_\theta} \ \star \ . \tag{37}$$

Hence, it can also be rewritten [24] as the evocative form,

$$e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x) + \hbar \overleftarrow{\partial}_\theta \overrightarrow{\partial}_\theta} \ . \tag{38}$$

Nevertheless, the original form displays associativity more readily, since the factor acting on the Grassmann structure is patently associative,

$$(A \diamond B) \diamond C = A \diamond (B \diamond C)\,, \tag{39}$$

acting on 1d bosonic superfields $A(\theta)=a+\theta\alpha$, $\ B(\theta)=b+\theta\beta$, so that

$$A \diamond B = ab + \hbar\alpha\beta + \theta(\alpha b + a\beta). \tag{40}$$

Note the loose analogy to complex multiplication $\overline{z}_1 z_2$. Even though this analogy cannot rise to an isomorphism, as evident from its noncommutativity and longer products such as the above, still, it turns out to be useful for actual evaluation of products in collecting the Grassmann even and odd terms in the answer. The symmetry of this product is further displayed by setting $\hbar=1$ and considering standard Grassmann Fourier transforms from bosonic to fermionic superfields, $\tilde{A}(\theta)=\int d\phi(1+\phi\theta)\ A(\phi)=\alpha+\theta a$:

$$A \diamond B = \tilde{A} \diamond \tilde{B}. \tag{41}$$

The first of refs [24] provides a diagonal extension to a space of more Grassmann variables ($N>1$ supersymmetry).

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